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Rigorous Derivation of Domain Growth Kinetics without Conservation Laws

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The time evolution of the Ising model that describes shrinking domains is studied. A singly connected domain of Ising spins, embedded in a sea of the opposite phase, develops at $T=0$ according to a dynamic rule that does not allow its perimeter to increase. At long enough times the domain disappears; we show that the average lifetime of such a domain is proportional to its area. We also consider the $T=0$ dynamics of a single infinite quadrant. The area of the quadrant decreases during the time evolution, and we show that the area lost grows linearly with time. We solve a first passage time problem as well. That is, we calculate the average time it takes for the area lost to reach a given value for the first time. Lastly, we map the infinite quadrant model onto a diffusion problem with exclusion in one dimension. This latter problem is mapped onto a critical six-vertex model.

KEY WORDS: Domain growth kinetics; Ising model; six-vertex model.

1. INTRODUCTION

Dynamics of systems quenched from a high temperature through an order–disorder transition exhibit universal features. Ordered domains appear in the initially disordered system following the quench. The shrinkage and growth of these domains at late stages of the evolution have been extensively investigated experimentally,[1] numerically,[2–5] and analytically.[6–8] Typically, it is found that the linear domain size L of the locally dominant phase grows with time as $L \sim t^x$. The exponent x depends on the system considered, in a way that is not completely understood. One important

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parameter that influences the growth kinetics is the nature of physical conservation laws. For example, the exponent x depends on whether the order parameter is conserved or not. In Ising like systems $x = 1/2$ when the order parameter is not conserved, while conservation of the order parameter slows the dynamics considerably. In that case it is widely accepted[5,9] that $x = 1/3$.

In this paper we present a rigorous calculation of x for a special case of the two-dimensional Ising model without conservation laws. We find $x = 1/2$, in accordance with earlier evidence. This result has been obtained analytically in a number of studies,[6–8] but to the best of our knowledge, it has not yet been demonstrated rigorously starting from a microscopic model.

The paper is organized as follows: First we review (in Section 2) some of the existing evidence for the $x = 1/2$ result, and motivate simplifications that will be introduced in order to enable a rigorous calculation. In Section 3 we present our model system and its solution. We also find a mapping between our model and a one-dimensional diffusion problem in which black and white particles occupy an infinite line; the dynamical moves are exchanges of black and white particles which are located at neighboring sites.[10] In the last part of Section 3 the diffusion problem is mapped onto a critical six-vertex model. Our results are summarized in Section 4.

2. EVIDENCE FOR $x = 1/2$ IN ISING-LIKE SYSTEMS

In this section we mention some of the important theoretical and experimental work on the subject. A more complete account of existing studies in the field can be found in review articles on dynamics near first-order phase transitions.[11]

Lifshitz[6] and later Cahn and Allen[7] (LCA) derived the $x = 1/2$ growth law starting from the Langevin equation associated with a time-dependent Ginzburg–Landau model:

$$\frac{\partial \phi(r, t)}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi(r, t)} + \xi(r, t)$$

This equation describes the evolution of the order parameter ϕ, and reflects the tendency to minimize the coarse-grained free-energy functional F during the evolution of the system. $\xi(r, t)$ is a Gaussian white noise source. LCA assumed that a typical configuration of the system at long times consists of large domains separated by narrow interfaces. In the bulk of each of the domains the order parameter takes one of its equilibrium values,

while at the interfaces it interpolates smoothly between these values. The dynamics at long times is then controlled by the evolution of the domain boundaries. One can write down a phenomenological evolution equation for a single boundary starting from the Langevin equation for the order parameter, and neglecting the noise term. LCA showed that the dynamics of such a boundary is driven by curvature, which readily gives the expected result. Valls and Mazenko[4] studied the growth kinetics of the time-dependent Ginzburg–Landau model mentioned above, by direct numerical solution of the associated Langevin equation *with* the noise term. Their results are consistent with the LCA theory. Similar results were obtained from renormalization group,[12] Monte Carlo renormalization group,[13] and finite-size scaling[5] calculations as well.

A very simple heuristic way to get the exponent x is to consider the shape of a domain of a minority phase surrounded by a sea of the majority phase. We expect the linear scale of the surrounding domain L to grow with time as $L \sim t^x$. Let us assume that the boundary that defines the minority domain is fairly circular, such that the radius of the domain is between R and $R + \delta R$, where $\delta R \ll R$. The system will show a tendency to decrease the surface free energy. As a result, the change of the domain volume V with time is determined by two competing effects: It will locally shrink at the convex parts of the interface (near the tips), but will grow at the concave regions (near the dips). The rate of increase (decrease) of V with time will be proportional to the surface area of the concave (convex) regions. Hence $\delta V/\delta t \sim -\delta R^{d-1}/\delta R$, where d is the dimensionality of the system. Using the relation $V \sim R^d$, we get

$$\delta R/\delta t \sim -1/R$$

The solution of this equation is

$$R_0^2 - R^2 \sim t$$

where R_0 is the initial radius of the shrinking domain. This gives rise to two equivalent results. First, that the lifetime of a shrinking domain of initial radius R_0 is proportional to R_0^2. Second, that the typical length scale of the majority domain, defined as $L \equiv (R_0^2 - R^2)^{1/2}$, grows as $L \sim t^{1/2}$, i.e., $x = 1/2$.

Experiments on Ising-like systems support the theoretical prediction outlined above. In a typical experiment one studies the ordering process in binary alloys, quenched from a high temperature to a temperature below the order–disorder transition. For example, Morris *et al.*, Hashimoto *et al.*, and Noda *et al.*[1] measured domain growth in Cu_3Au alloy, by X-ray scattering and electron microscopy. They found that the average domain size grows as $t^{1/2}$.

These binary alloys are modeled as Ising antiferromagnets. Phani *et al.*, Sahni *et al.*, and Kaski *et al.*[2] performed simulations of this model in two and three dimensions, using spin-exchange Kawasaki dynamics.[14] Although the total magnetization is then conserved, the (antiferromagnetic) order parameter is not, and their results agree with the experiments and the theoretical predictions. Sahni *et al.*[3] simulated the same model using Glauber single-spin-flip dynamics[15] as well. They found the same qualitative result.

The ferromagnetic Ising model exhibits the same behavior.[16] In Fig. 1 we show results of simulations of this model on a 64×64 square lattice with periodic boundary conditions. Glauber dynamics was used. The system was simulated at $T = 0.7T_c$ (T_c being the infinite lattice critical temperature), starting from a random initial state (each of the spins was assigned a value of ± 1 with probability 1/2). Snapshots of the system at later times are shown, with regions of $+$ ($-$) spins drawn in black (white). Following the evolution of the system, we see that at long times there is a single large black domain surrounded by a white sea. There are small black

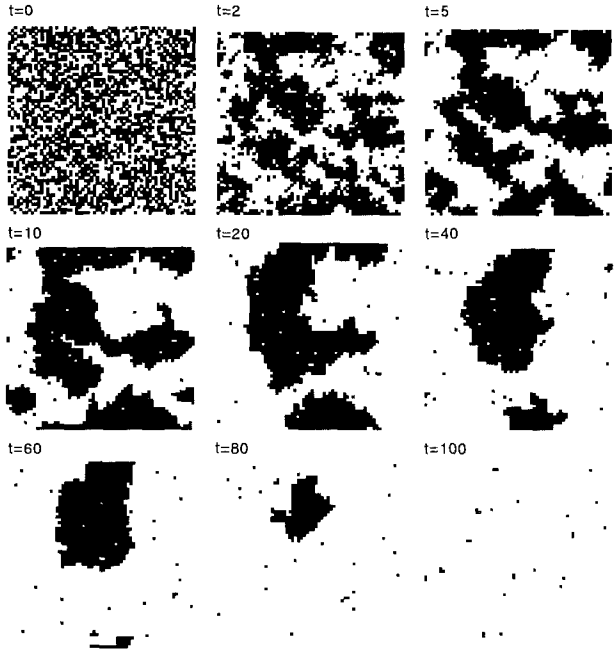

Fig. 1. Simulation of a 64×64 Ising lattice at $T = 0.7T_c$, starting from a random initial condition. Time is measured in Metropolis sweeps. Periodic boundary conditions were used; the structure seen at $t \geqslant 20$ is therefore a single domain. At $t = 100$ the domain has disappeared and characteristic equilibrium fluctuations are present.

regions inside the sea, whose size is of the order of the average equilibrium size of the minority phase at $T=0.7T_c$. The black domain then shrinks until it reaches this average equilibrium size. Thus, we are motivated to make a simplifying assumption, that at long times we have to deal with the shrinkage of a single domain of the minority phase in a sea of the dominant phase. We will argue in the next section that the physics at finite $T<T_c$ is the same as at $T=0$. The $T=0$ dynamics of the single domain reduces to a one-dimensional problem which is easier to solve. Such contour dynamics at finite temperature were studied by Sokal and Thomas.[17]

3. DYNAMICS OF A SINGLE DOMAIN AT $T=0$

We first define the model we solved and present the solution, e.g., calculate the *average lifetime of a domain at* $T=0$. We also solve a problem in which the finite domain of opposite spins is replaced by an infinite quadrant. Finally, this problem is mapped onto a many-particle diffusion problem in one dimension, which, in turn, is equivalent to a critical six-vertex model.

3.1. Finite Domain Model: Definition and Solution

We turn now to study the manner in which a single finite domain of, say, (+) spins, in an infinite sea of (−), develops in time. Numerical simulations have established[3] that for large domains the *area decreases linearly with time*, i.e..

$$A=A_0-\alpha t \tag{1}$$

Obviously, this relationship implies that the lifetime of a domain is proportional to its area. This linear dependence holds at subcritical temperatures $T<T_c$, including $T=0$. Thus, the physics of shrinking domains is, apparently, the same.

Let us present the results of simulations performed at $T=0$, on an initial state that contained a 60×60 square domain, in a sea of opposite spins. In the course of the simulation a spin is selected at random, and an attempt to flip it is made. This "move" is accepted if it does not result in an increase of the energy. The energy is equal to the number of broken bonds. Only those moves will be accepted that either leave this number invariant or reduce it. Thus, the only possible moves involve spins on either side of the domain boundary. The dynamics is therefore restricted to changing the contour of the domain in a manner that either reduces its length or leaves it constant. Initially only one of the four corner spins of the

domain can flip. Such a flip creates steps on the vertical and horizontal boundaries of the domain. These steps are free to move back and forth with equal probability. This diffusive motion of the steps does not change the length of the domain boundary, and hence the energy remains unchanged. On the other hand, when two steps on the same boundary approach one another from opposite sides, they may annihilate, thereby reducing the energy. This last process is irreversible at $T=0$. Typical configurations, as obtained at various times, are shown in Fig. 2. We also show, in Fig. 3, the area A as a function of time t, measured in units of "sweeps"; in a single time step the number of attempted flips equals the total number of spins in the system. Figure 3a presents $A(t)$ for a single simulation, whereas in Fig. 3b we show the result of averaging over 20 systems developing from the same initial condition. The linear dependence on t is clearly seen; deviations at late times are due to finite-size effects that take over when the domain size is small.

The dynamic problem discussed above is still not soluble analytically. Therefore we simplify it further by considering the same dynamic problem with mirror boundary conditions. That is, we perform the same simulation procedure as described above, but the spin to be flipped is selected from a

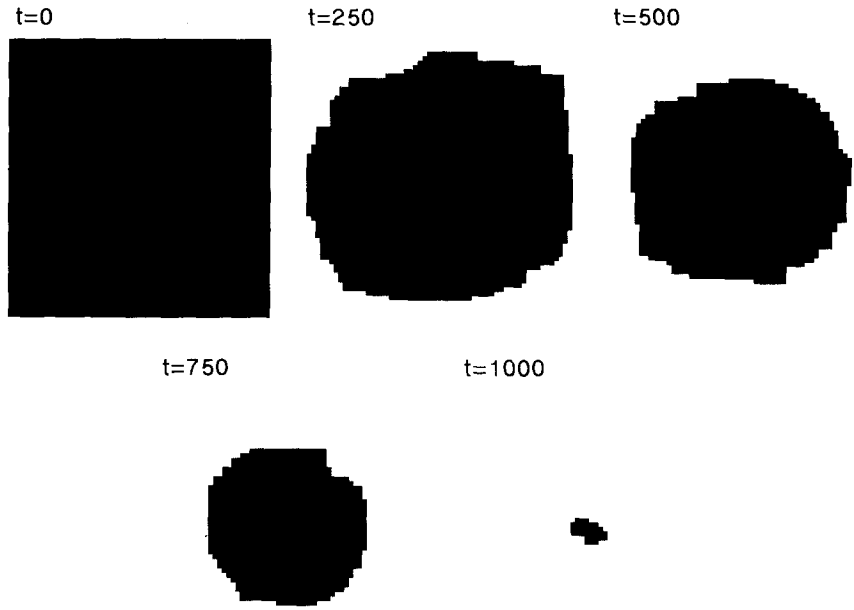

Fig. 2. A finite square domain embedded in an infinite sea of opposite spins shrinks at $T=0$. The physics is the same at nonzero temperature as at $T=0$; and since the latter case is more readily solvable analytically, we focus our attention on it.

single quadrant of the plane, and then if it is flipped, its three images on the three other quadrants are also flipped. These images are obtained by reflections with respect to mirror planes placed on the x and y axes. Thus, we follow the domain in a single quadrant only. Whenever a step that moves along a boundary reaches a mirror, it is annihilated, since, due to the boundary conditions, an opposite "antistep" is reaching the mirror at the same time.

Contours obtained in a typical run are shown in Fig. 4; the initial quadrant contained a 30×30 domain. The area versus time, averaged over

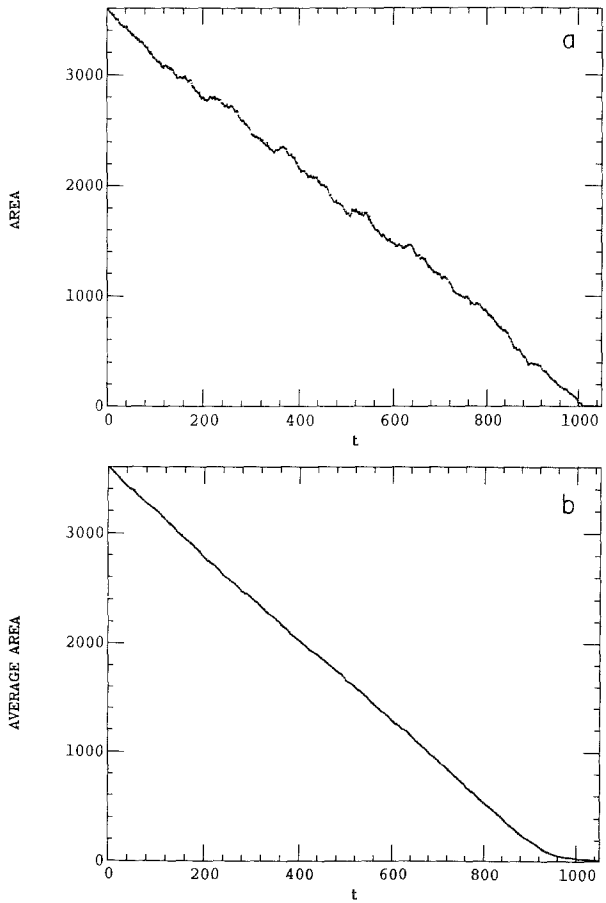

Fig. 3. (a) The area of the shrinking domain of Fig. 2, as a function of time. When 20 simulations (starting from the same initial state) are averaged, the smooth linear function (b) is obtained.

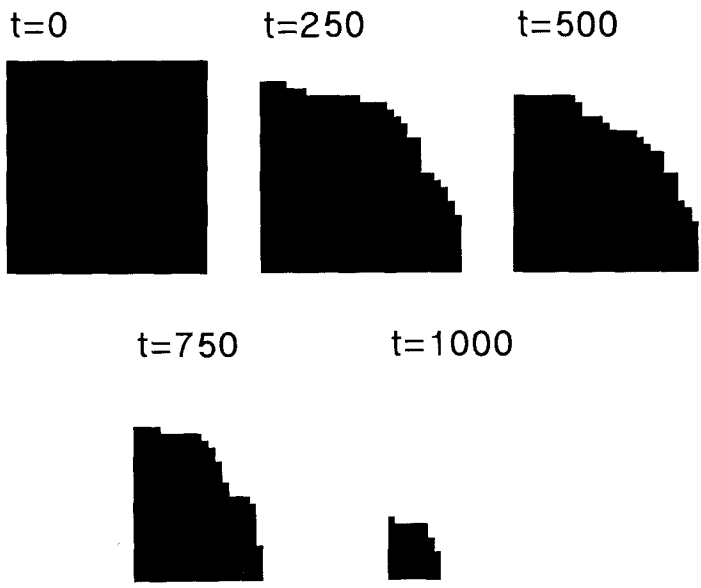

Fig. 4. Typical simulation of a single quadrant, with the boundary conditions discussed in the text.

60 runs, is shown in Fig. 5. The linear law is again evident, and the resemblance to Fig. 3 indicates that the physical origin of this law has not been lost by working with a single quadrant.

The quantity we calculated for this problem is τ_A, *the average death time of a domain of initial area A*. That is, we calculate how many time steps are needed until the domain (quadrant) of initial area A shrinks completely and disappears. In order to calculate this quantity we first map the dynamics of the contour as described above onto a random walk problem, for which a classical ruin problem (see, e.g., ref. 18) (average duration of the game) is solved.

To see how this mapping comes about, consider Fig. 6, which presents a typical contour obtained from an initial state of an $L \times N$ quadrant. We associate with each horizontal row j a walker whose position l_j marks the rightmost edge of the contour on that row. At all times the walkers remain *ordered*:

$$0 \leqslant l_N \leqslant l_{N-1} \leqslant \cdots \leqslant l_2 \leqslant l_1 \tag{2}$$

The dynamics of our domain is reproduced by the following rules:

1. Randomly pick one walker j (out of N).
2. If $l_j = 0$, do not move

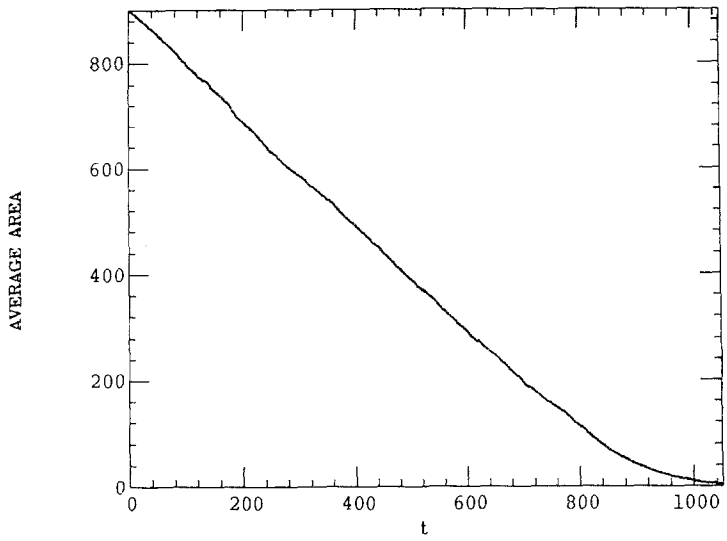

Fig. 5. Area of a shrinking finite quadrant, averaged over 60 runs.

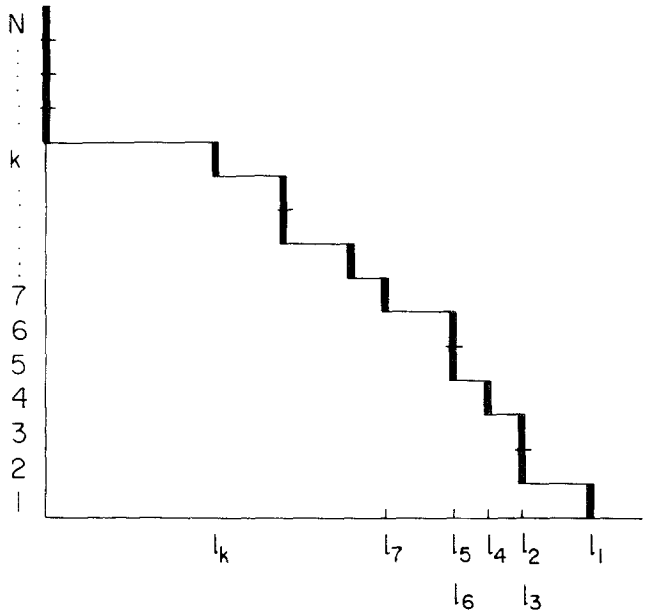

Fig. 6. Mapping the shrinking quadrant problem to one of N random walkers. Each walker is represented by a heavy vertical line segment, allowed to walk horizontally. The walkers are indexed on the left side of the vertical axis. Positions of the walkers are indicated below the horizontal axis. The figure shows a snapshot taken after walkers $k+1,\ldots, N$ have arrived at $l=0$, and are "stuck" there.

3. If $l_j > 0$, with probability $0 < p \leqslant 1/2$ *attempt* to move walker j to $l_j + 1$, or with same probability to $l_j - 1$. Note that with probability $1 - 2p$ no attempt is made to move the walker.
4. An attempt is successful if the new position does not violate the ordering, Eq. (2). If the attempt failed, the walker stays at l_j.
5. The first walker ($j = 1$) is not allowed to move to the right (i.e., l_1 can only decrease).

These rules reproduce precisely the dynamics of the shrinking domain on one quadrant, with one minor modification: The basic time step here involves an attempt to flip the "right" spin in each row, and hence N of these steps are equivalent to NL spin-flip attempts that constitute a single Monte Carlo sweep of the spin system. Also, allowing $p < 1/2$ changes the result by a trivial rescaling of time.

We calculate $\tau(l_1, l_2, l_3, \ldots, l_k, 0, 0, \ldots, 0)$, the average time it takes for *all* the walkers to reach the origin, starting from an initial state in which walkers $j = 1, 2, \ldots, k$ are at positions $l_j > 0$, and walkers $j = k+1, k+2, \ldots, N$ at $l_j = 0$. This average ruin time is calculated by writing down a recursion for it. To compress notation, we omit the 0's from the argument of τ. The recursion takes the form

$$\begin{aligned}
\tau(l_1, \ldots, l_k) = {} & \frac{N-k}{N} [\tau(l_1, l_2, \ldots, l_k) + 1] \\
& + \frac{1-p}{N} [\tau(l_1, l_2, \ldots, l_k) + 1] \\
& + \frac{p}{N} [\tau(\max(l_1 - 1, l_2), l_2, \ldots, l_k) + 1] \\
& + \frac{1-2p}{N} \sum_{j=2}^{k} [\tau(l_1, l_2, \ldots, l_k) + 1)] \\
& + \frac{p}{N} \sum_{j=2}^{k} [\tau(l_1, l_2, \ldots, \min(l_j + 1, l_{j-1}), \ldots, l_k) + 1] \\
& + \frac{p}{N} \sum_{j=2}^{k} [\tau(l_1, l_2, \ldots, \max(l_j - 1, l_{j+1}), \ldots, l_k) + 1] \qquad (3)
\end{aligned}$$

The first term represents the events of picking one of the $N-k$ walkers that have already arrived at $l_j = 0$. The second and third terms arise from choosing walker number 1. The second term represents not moving it, or attempting to move it right. Since it cannot move right, the second term contains $\tau(l_1, l_2, \ldots, l_k)$, whereas the third term, which represents an attempt

to move it left, allows the replacement $l_1 \to l_1 - 1$, provided $l_1 - 1 \geqslant l_2$. The next term corresponds to choosing one of walkers $j = 2, 3, \ldots, k$, but deciding not to move it. The last two terms describe attempts to move walkers $j = 2, 3, \ldots, k$ right and left, again in a manner that ensures that the ordering of Eq. (2) is not violated.

After some algebra this equation can be rewritten as

$$\begin{aligned}(2k-1)\,\tau(l_1, l_2, \ldots, l_k) &= \tau(l_1, l_2, \ldots, l_k - 1) \\ &\quad + \sum_{j=2}^{k} \tau(l_1, l_2, \ldots, \min(l_j + 1, l_{j-1}), \ldots, l_k) \\ &\quad + \sum_{j=1}^{k-1} \tau(l_1, l_2, \ldots, \max(l_j - 1, l_{j+1}), \ldots, l_k) + \frac{N}{p} \end{aligned} \tag{4}$$

At this point we guess the solution

$$\tau(l_1, l_2, \ldots, l_k) = \frac{N}{p} \sum_{j=1}^{k} l_j \tag{5}$$

Substituting (5) easily verifies that indeed it solves (4). The boundary condition $\tau(0, 0, \ldots, 0) = 0$ is also satisfied.

Obviously, when properly normalized (i.e., divided by N; see above) our solution has the form

$$\tau(l_1, l_2, \ldots) = \frac{1}{p} A(l_1, l_2, \ldots) \tag{6}$$

where A is the *area* of the domain. Hence we have demonstrated that *the average lifetime of a domain is proportional to its area.*

This result is not completely obvious to interpret in terms of our model of N diffusing particles. The average lifetime of a single particle that gets annihilated at distance L from the origin is proportional to L^2. In the present problem the hard core repulsion between neighboring particles and the fact that the last one could only move in one direction change this behavior to NL.

3.2. Dynamics of an Infinite Quadrant

We now turn to describe the solution of another, closely related problem. Consider an initial state of an *infinite system*, in which in three quadrants of the plane the spins are in state $(-)$ and in the fourth

quadrant in the state (+). At $T=0$ such an initial state will develop in time in a manner that leaves invariant the length of the (infinite) contour of this "domain." Initially only the single corner spin can flip, but at later times we can have again many steps running along both horizontal and vertical boundaries. A typical configuration of the contour is shown in Fig. 7. For this problem there is no irreversible step, as we had before, that reduces the area of the domain. Nevertheless, if we consider A_l, the area "lost" as a function of time, we note that it executes a (complex) random walk with an infinite barrier at $A_l=0$. Hence we expect the average A_l to increase with time. Figure 6 represented the finite domain dynamics with mirror plane symmetry. It may also represent the infinite quadrant problem; the area lost A_l will replace the area of the finite domain, and the walkers will tend to move to the right rather than to the left.

The problem of the infinite quadrant can be viewed as the limit of finite problems with increasing number of walkers N. We will first define the problem with a finite number of walkers, then solve it and take the limit $N\to\infty$. Consider, as in the finite domain case, N walkers, which remain ordered at all times according to Eq. (2). In the finite domain problem we had a sink for walkers at $l=0$, and the first walker was not allowed to move to the right. The new problem is different in that the sink

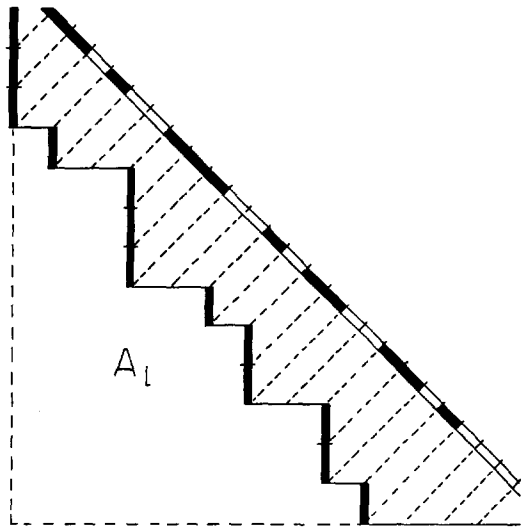

Fig. 7. The area lost A_l at the corner of an infinite shrinking quadrant. Vertical and horizontal edges of the domain's boundary are projected onto a diagonal line, giving rise to an assignment of black and white particles. Every change of the domain's contour maps onto an exchange of neighboring particles of different colors.

is eliminated, and the condition on the first walker is replaced by a restriction on the Nth walker, which is not allowed to move to the left. The dynamic rules for the new problem can be summarized as follows:

1. Randomly pick one walker j (out of N).
2. With probability $0 < p \leqslant 1/2$, *attempt* to move walker j to $l_j + 1$, or with same probability, to $l_j - 1$. With probability $1 - 2p$, no attempt is made to move the walker.
3. An attempt is successful if the new position does not violate the ordering, Eq. (2). If the attempt failed, the walker stays at l_j.
4. The Nth walker ($j = N$) is not allowed to move to the left (i.e., l_N can only increase).

Although this finite domain problem has an irreversible step (rule 4), it does approach the infinite domain model in the limit $N \to \infty$. In order to see that the irreversible step does not destroy the limit, consider the average time $\tau_0(j)$ it takes until walker j ($j = 1, 2, \ldots, N$) performs its first move. The irreversibility of moves of the Nth walker affects the dynamics only at times $t > \tau_0(N)$. We will show that after scaling $\tau_0(N)$ by N, it still diverges as $N \to \infty$, and hence as N is increased, the finite domain problem approaches the infinite quadrant dynamics.

To prove that $\tau_0(N)$ diverges, let us define a simpler model in which *all* the walkers can move only to the right. $\tau_0(j)$ is clearly bounded from below by $\bar{\tau}_0(j)$, the average time of the first move of walker j in the simpler model. From the definition of this model we have

$$\bar{\tau}_0(j) = \bar{\tau}_0(j-1) + \bar{\tau}_0(1) \tag{7}$$

since the jth walker cannot move before walker $j-1$ has moved. After the move of walker $j-1$, the jth walker is not restricted (at least until after its first move), and hence it will wait $\bar{\tau}_0(1)$ time units before moving. Iterating Eq. (7) down to the first walker, we get

$$\bar{\tau}_0(j) = j\bar{\tau}_0(1) \tag{8}$$

$\bar{\tau}_0(1)$ can be calculated exactly:

$$\begin{aligned}\bar{\tau}_0(1) &= 1 \cdot \frac{p}{N} + 2 \cdot \left(1 - \frac{p}{N}\right)\frac{p}{N} + 3 \cdot \left(1 - \frac{p}{N}\right)^2 \frac{p}{N} + \cdots \\ &= \frac{p}{N} \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left(1 - \frac{p}{N}\right)^i = \frac{p}{N} \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \left(1 - \frac{p}{N}\right)^i = \frac{N}{p}\end{aligned} \tag{9}$$

Using Eq. (8) in conjunction with Eq. (9) and scaling time by a factor of N, we get

$$\tau_0(j) \geqslant \bar{\tau}_0(j) = \frac{j}{p} \tag{10}$$

This implies that $\tau_0(N) \geqslant N/p$, and therefore irreversible steps are not allowed in the limit $N \to \infty$.

In what follows we calculate two quantities of the finite domain problem. The first one is the first passage time $\tau_A^l(l_1, l_2, \ldots, l_N)$, the average time it takes for the area lost to reach the value A for the first time, starting from the initial configuration $(l_1, l_2, \ldots, l_N)$ of the walkers, where the area lost is defined as

$$A_l(t) \equiv \sum_{j=1}^{N} l_j(t) \tag{11}$$

The second quantity is the average area lost as a function of time, $\langle A_l(t) \rangle$.

To calculate the first passage time, we use the same considerations that led to Eqs. (3) and (4). Equation (3) is now replaced by

$$\begin{aligned}
&\tau_A^l(l_1, l_2, \ldots, l_N) \\
&\quad = \frac{1-2p}{N} \sum_{j=1}^{N} [\tau_A^l(l_1, l_2, \ldots, l_N) + 1] + \frac{p}{N} [\tau_A^l(l_1 + 1, l_2, \ldots, l_N) + 1] \\
&\qquad + \frac{p}{N} \sum_{j=2}^{N} [\tau_A^l(l_1, l_2, \ldots, \min(l_j + 1, l_{j-1}), \ldots, l_N) + 1] \\
&\qquad + \frac{p}{N} \sum_{j=1}^{N-1} [\tau_A^l(l_1, l_2, \ldots, \max(l_j - 1, l_{j+1}), \ldots, l_N) + 1] \\
&\qquad + \frac{p}{N} [\tau_A^l(l_1, l_2, \ldots, l_N) + 1]
\end{aligned} \tag{12}$$

The first term represents the events of picking one of the N walkers, but deciding not to move it. The second term describes attempts to move the first walker to the right. This move is always allowed, and hence the term contains $\tau_A^l(l_1 + 1, l_2, \ldots, l_N)$. The third term arises from choosing one of walkers $j = 2, 3, \ldots, N$ and attempting to move it right. The move is allowed if $l_j + 1 \leqslant l_{j-1}$. The next term, which represents an attempt to move one of the walkers $j = 1, 2, \ldots, N-1$ to the left, allows the replacement $l_j \to l_j - 1$, provided $l_j - 1 \geqslant l_{j+1}$. The last term corresponds to an attempt to move the Nth walker to the left. Since this move is not allowed, the term contains $\tau_A^l(l_1, l_2, \ldots, l_N)$.

Algebraic manipulations of Eq. (12) lead to the simplified equation

$$\begin{aligned}(2N-1)\,\tau_A^l(l_1, l_2,..., l_N) &= \tau_A^l(l_1+1, l_2,..., l_N) \\ &+ \sum_{j=2}^{N} \tau_A^l(l_1, l_2,..., \min(l_j+1, l_{j-1}),..., l_N) \\ &+ \sum_{j=1}^{N-1} \tau_A^l(l_1, l_2,..., \max(l_j-1, l_{j+1}),..., l_N) + \frac{N}{p}\end{aligned} \tag{13}$$

As in the finite domain problem, we guess the solution

$$\tau_A^l(l_1, l_2,..., l_N) = \frac{N}{p}\left(A - \sum_{j=1}^{N} l_j\right) \tag{14}$$

It is easy to verify that (14) indeed solves Eq. (13). The solution is valid only for initial configurations which satisfy the relation $\sum_{j=1}^{N} l_j \leqslant A$, and satisfies the boundary condition $\tau_A^l(l_1, l_2,..., l_N) = 0$ whenever $\sum_{j=1}^{N} l_j = A$. Scaling time units by N, and setting the initial configuration to $l_j = 0$ for $j = 1, 2,..., N$, we get

$$\tau_A^l = \frac{1}{p} A \tag{15}$$

We turn now to discuss the dependence of the average area lost $\langle A_l(t)\rangle$ on time. In order to do this, let us introduce $W(f(\{l_j\}); t)$, the probability that the condition $f(\{l_j\})$ is satisfied at time t. For example, $W(l_j = l_{j+1}; t)$ is the probability that walkers j and $j+1$ are at the same position at time t. Our strategy will be to find equations for $W(l_j = l; t)$, and then use them to calculate

$$\langle A_l(t)\rangle = \sum_{j=1}^{N} \sum_{l} l W(l_j = l; t) \tag{16}$$

Our dynamic rules allow us to express $W(l_j = l; t+1)$ in terms of probabilities at an earlier time t:

$$\begin{aligned}W(l_1 = l; t+1) = W(l_1 = l; t) &- \frac{p}{N} W(l_1 = l, l_2 \neq l; t) \\ &- \frac{p}{N} W(l_1 = l; t) + \frac{p}{N} W(l_1 = l+1, l_2 \neq l+1; t) \\ &+ \frac{p}{N} W(l_1 = l-1; t)\end{aligned}$$

$$W(l_j = l; t+1) = W(l_j = l; t) - \frac{p}{N} W(l_j = l, l_{j+1} \neq l; t)$$

$$- \frac{p}{N} W(l_j = l, l_{j-1} \neq l; t)$$

$$+ \frac{p}{N} W(l_j = l+1, l_{j+1} \neq l+1; t) \qquad \text{for} \quad 2 \leqslant j \leqslant N-1$$

$$+ \frac{p}{N} W(l_j = l-1, l_{j-1} \neq l-1; t)$$

$$W(l_N = l; t+1) = W(l_N = l; t) - \frac{p}{N} W(l_N = l, l_{N-1} \neq l; t)$$

$$+ \frac{p}{N} W(l_N = l-1, l_{N-1} \neq l-1; t) \tag{17}$$

The negative terms correspond to situations where walker j was at position l at time t, but moved from there at time $t+1$, while positive terms represent cases where walker j moved to position l at time $t+1$. Multiplying Eq. (17) by l and summing over all possible values of l, we obtain equations for $\langle l_j(t) \rangle$, the average position of walker j at time t:

$$\langle l_1(t+1) \rangle = \langle l_1(t) \rangle + \frac{p}{N} W(l_1 = l_2; t)$$

$$\langle l_j(t+1) \rangle = \langle l_j(t) \rangle + \frac{p}{N} W(l_j = l_{j+1}; t)$$

$$- \frac{p}{N} W(l_j = l_{j-1}; t) \qquad \text{for} \quad 2 \leqslant j \leqslant N-1 \tag{18}$$

$$\langle l_N(t+1) \rangle = \langle l_N(t) \rangle + \frac{p}{N} - \frac{p}{N} W(l_N = l_{N-1}; t)$$

In order to derive Eq. (18), we used the identities $W(l_j = l,\ l_{j\pm1} \neq l; t) + W(l_j = l, l_{j\pm1} = l; t) = W(l_j = l; t)$ and $\sum_l W(l_j = l; t) = 1$.

When Eq. (18) is summed over all walkers, all the two-particle correlation terms cancel and we get

$$\langle A_l(t+1) \rangle = \sum_{j=1}^{N} \langle l_j(t) \rangle = \langle A_l(t) \rangle + \frac{p}{N} \tag{19}$$

Equation (19) can be used recursively to express $\langle A_l(t) \rangle$ in terms of $\langle A_l(0) \rangle$. Scaling time units by N, we obtain the final result:

$$\langle A_l(t) \rangle = \langle A_l(0) \rangle + pt \tag{20}$$

It was pointed out to us by Eugene R. Speer that this result can be obtained from a very simple argument. In any given configuration (see Fig. 6) only spins at corners of the interface can flip. There are two kinds of such corners, convex and concave. When a spin which is located at a convex corner is flipped, the area lost A_l is reduced by 1, while a flip of a spin that sits at a concave corner increases A_l by 1. Since such flips occur with probability p per unit time, the rate of change of A_l obeys the equation

$$\frac{d\langle A_l(t)\rangle}{dt} = p(N_{\text{concave}} - N_{\text{convex}}) \tag{21}$$

where N_{convex} (N_{concave}) is the number of convex (concave) corners (which depends on time). Clearly, in any allowed configuration there is an excess of exactly one concave corner, and hence Eq. (21) reduces to

$$\frac{d\langle A_l(t)\rangle}{dt} = p \tag{22}$$

which readily leads to Eq. (20). This argument does not hold in the finite domain problem with the sink. It fails for long enough times, when the configuration $l_j = 0$ for $j = 1,..., N$ appears with finite probability. In this configuration, Eq. (22) is not valid, and hence the argument breaks down.

Since both of our results [Eqs. (14) and (20)] do not depend on N, they are valid in the limit of an infinite quadrant. In spite of the simplicity of these results, the time evolution of the system is very complicated, as can be seen from the behavior of the position of a single walker with time. This behavior is completely different in the finite- and infinite-N problems. Consider the difference $D_j(t) \equiv \langle l_{j+1}(t)\rangle - \langle l_j(t)\rangle$. In the case $N = 2$ this variable performs a simple random walk with a reflecting barrier at 0. Therefore, as $t \to \infty$, D diverges as $\sqrt{t}$. The presence of other walkers can only slow down the increase of D with time, and hence

$$\lim_{t \to \infty} \frac{D_j(t)}{t} = 0 \tag{23}$$

In order that the area lost will grow linearly with time, each of the walkers must have a finite velocity in the long-time limit. Combining this observation with Eq. (23), we conclude that in this limit all the walkers move with the same velocity,

$$v = p/N \tag{24}$$

This finite velocity is a consequence of the irreversible steps in the finite domain dynamics. As expected from the fact that there is no irreversibility

in the infinite-N limit, the velocity v vanishes in this limit, leading to a completely different dependence of the position of a single walker on time. However, the result for the area lost still holds, and implies that in the long-time limit $\langle l_j(t) \rangle$ grows as $\sqrt{t}$, rather than linearly. At any given time t, the average number of walkers that have already started to move grows as $\sqrt{t}$ as well, leading to the desired linear dependence of A_l on time.

Similar infinite quadrant models were solved by Rost[19] and Marchand and Martin,[20] who studied the dynamics of domain boundaries in the presence of magnetic fields. Their solutions are not valid for the case of no external field, which is the problem that we considered.

3.3. Mapping onto Diffusion in One Dimension, and the six-Vertex Model

It is interesting to note that the last problem, of a shrinking infinite quadrant, maps onto a fairly widely studied[10] many-body diffusion problem in one dimension. Namely, consider a case of black and white particles arranged initially so that all sites $x > 0$ are occupied by black and all sites with $x < 0$ by white particles. Now allow a diffusion process in which exchange of black and white particles that occupy neighboring sites is the basic move. That is, pick a pair of neighboring sites, and exchange, with probability p, the two particles that occupy the sites picked. If the two particles are of the same color, such an exchange has no effect on the particle configuration; only exchange of black and white particles matters. An alternative interpretation of the same problem views the black particles as "particles" and the white ones as "holes." With this interpretation exchange of a neighboring pair of different colors is viewed as hopping of a particle onto an empty site.

Figure 7 demonstrates how every configuration of the contour of the shrinking quadrant described above is mapped to an assignment of particles on a line. The particle configuration is obtained by projecting the contour onto the diagonal line, as demonstrated. Vertical (horizontal) edges of the contour become black (white) particles. Furthermore, the dynamic rules we used for the shrinking single-quadrant problem reproduce the diffusive process of these particles. It is straightforward to see that the "area lost" in the previous problem equals the total net number of steps taken by black particles in the right (x) direction.

The diffusive process described above treats a single pair of sites at a time. It is easy to define a closely related problem, in which pairs are treated in parallel, and which has the same physics. To do this, pair all neighboring sites: This can be done in two ways (denoted A and B), with

either the even- or the odd-indexed sites at the left side of every pair (see Fig. 8a). We choose pairing A at even time steps and B at odd time steps. At each time step, attempts to exchange the particles within the appropriate pairs are allowed.[21] The diffusion problem is now represented as a one-dimensional probabilistic cellular automaton. The time development of such an automaton is equivalent to an equilibrium problem in $d=2$ dimensions,[22] which we now identify as a particular case of the six-vertex model.

To see how this mapping comes about, consider Fig. 8. The particle configuration present at some time step is depicted on horizontal lines, with (black and white) particles represented by arrows. Time is running upward; the state of the system at the next time step is represented by the next horizontal row, above the present one (see Fig. 8b). To describe the dynamics, assign to each site an edge, or line segment, whose center is placed at the corresponding site. The orientation of each edge reflects the

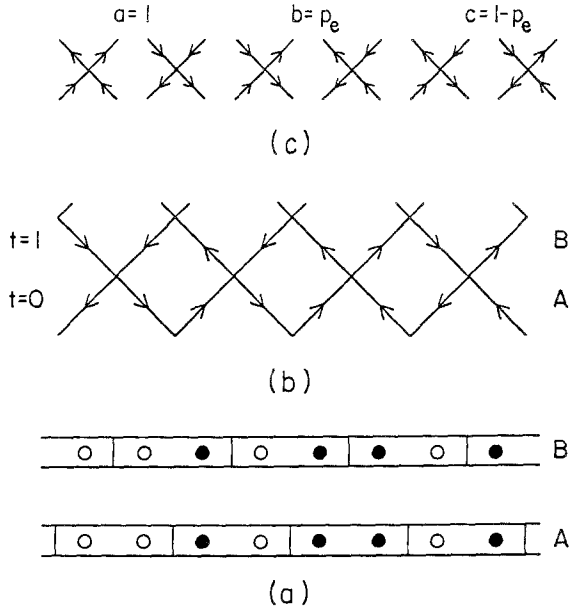

Fig. 8 (a) Two different partitionings, A and B, of a linear chain, occupied by black and white particles, into paired neighboring sites. (b) Representing black (white) particles by up- (down-) pointing arrows. Every horizontal cut presents a snapshot of the particles at a given time step. The arrows are placed on line segments; segments associated with paired neighboring sites meet at a vertex *above* the current time step. (c) At each vertex an exchange of particles may take place, giving rise to a new configuration. The probabilities of all possible processes are identified as the standard weights of the corresponding six-vertex model.

pairing (A or B) of the present time step, in that the edges running through paired sites meet *above* the current time slice. Evidently this representation automatically reproduces the change of pairing of neighboring sites after each time step. Pairs of edges meet at a *vertex*, at which exchange of particles may take place. A black particle is represented by an arrow that points up (and left or right), whereas white particles by arrows that point down (and right or left). Two edges emerge from each vertex; the centers of these edges are again at the sites of our (one-dimensional) lattice of diffusing particles. The state of the arrows at the centers of these emerging edges represents the particle configuration at the next time step.

The new particle or arrow configuration is the result of "collisions," or particle exchanges, that took place at each vertex. In order to reproduce the dynamics of our diffusion problem, we choose the following rules for these collisions:

1. If the two particles incident on a vertex are the same (both arrows up or both down), the emergent state is identical to the incident one.
2. If the two particles are different, they are either exchanged, with probability p_e, or not, with probability $1-p_e$.

These rules reproduce precisely the diffusion problem with parallel dynamics. On the other hand, the space-time history of the particle system is now mapped onto a configuration of arrows on the edges of a square lattice. It is easy to see that the rules listed above give rise to six allowed vertices, with nonvanishing weights, shown in Fig. 8c. The probability of observing any configuration of arrows (or the corresponding space-time history of the diffusing particles) is given, as in the standard six-vertex model, by the product of the weights of all the vertices in the configuration. It is interesting to note that our diffusion problem maps onto a *critical* six-vertex model.[23] Solution of this vertex model, and its implications for the diffusion problem, will be presented elsewhere.[24]

4. SUMMARY

We considered the dynamics of an Ising system at its coexistence ($H=0$) line, below the transition point. We argued that the dynamics with no conserved quantity is governed by the manner in which finite domains shrink. This problem, in turn, can be studied at zero temperature without losing the important physical property of length scaling as the square root of time,

$$L \sim t^{1/2}$$

This relationship can also be viewed as a statement on the lifetimes of finite domains, e.g., that the lifetime should be proportional to the domains' area. We have derived such a relationship rigorously. We also solved the time evolution of an infinite quadrant. We have shown that the area lost by the quadrant grows linearly with time, and that the time it takes for the area lost to reach the value A for the first time is proportional to A. Furthermore, we mapped the infinite quadrant problem at $T=0$ onto a many-body diffusion problem in one dimension, which in turn can be mapped onto a critical six-vertex model.

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